MANDELBROJT'S INEQUALITY AND DIRICHLET SERIES WITH COMPLEX EXPONENTS

BY

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1. **Introduction.** We consider Dirichlet series

$$\sum_{n=1}^{\infty} c_n e^{-\lambda_n z}, \tag{1.1}$$

where the exponents $\{\lambda_n\}$ form a sequence of complex numbers having arbitrary arguments but increasing absolute values

$$0 < |\lambda_1| < |\lambda_2| < \cdots \tag{1.2}$$

and a finite upper density

$$D^* = \limsup_{n\to\infty} \frac{n}{|\lambda_n|} < \infty. \tag{1.3}$$

We shall, at times, assume that there exists a positive number p such that

$$|\lambda_{n+1}| - |\lambda_n| \geqq p > 0 \tag{1.4}$$

for $n=1, 2, \cdots$, which implies (1.3) with $D^* \leqq 1/p$. Dirichlet series with complex exponents have been investigated by Ritt [18], [19], Väisälä [22], Hille [5], Pólya [16], Valiron [23], Leont'ev [7], Kahane [6], and the author [4].

Mandelbrojt's inequality [9, p. 77] gives an estimate of the coefficient c_n in terms of the maximum modulus of the analytic continuation of a function $f(z)$ to which the series converges (or which it merely represents asymptotically; this is of the greatest significance, although here we shall consider only convergent series). This generalization of Cauchy's estimate of the coefficients of a Taylor series has proved very fruitful in a number of seemingly diverse areas of the theory of functions [8; 9], one of which is the detection of singularities. However, the usual formulations of the inequality do not lead to the most delicate possible results in this last area of investigation, unless $D^* = 0$, because one can not "approach closer" than $\pi\overline{D}^*$ to a singularity of $f(z)$, where $\overline{D}^*$ is the mean upper density (see §3 below) of the sequence $\{\lambda_n\}$. To be more precise: it has usually been assumed that the series represents $f(z)$ in a domain containing a disc of radius $\pi\overline{D}^*$, and that the analytic continuation of $f(z)$ is along a curve each point of which is the center of a disc of radius $\pi\overline{D}^*$ on which the continuation of $f(z)$ is regular. It will be

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shown here, however, that this disc of radius $\pi\overline{D}^*$ can be replaced by the reflection with respect to the imaginary axis of the indicator diagram, a much used concept due to Pólya [14; 17], of any entire function of exponential type having simple zeros at the points $\lambda_1, \lambda_2, \cdots$. This entire function can be chosen so that its indicator diagram is at least as small as, and in most cases much smaller than, the disc. For example, in the special case of positive exponents which have a density

$$D = \lim_{n\to\infty} \frac{n}{|\lambda_n|}, \tag{1.5}$$

the disc of radius $\pi\overline{D}^* = \pi D$ can be replaced by a vertical line segment of length $2\pi D$. By means of this modification we shall achieve arbitrarily close approach to certain singular points of $f(z)$.

Applications of our modification of Mandelbrojt's inequality begin with a convergence theorem. An example is given which shows an obvious need for sufficient conditions to assure uniform convergence on an unbounded set; and our theorem meets this requirement; but of much more interest is the description, given in this theorem, of a set on which the series (1.1) converges. Leont'ev [7, p. 80] has described a similar set, but it is smaller, because his method of proof employs discs of radius πD^* rather than the reflected indicator diagram mentioned above (except in the particular case of positive or "nearly positive" exponents). Kahane [6, pp. 85, 92] gives related results in a much more general setting than ours, and he makes systematic use of the indicator diagram; but he gives specific conditions to assure convergence of a series only when he has placed restrictions on the arguments of the exponents [6, pp. 93–99].

Regions which contain singular points for the sum of the series are then discovered, as an immediate consequence of our convergence theorem. The principal result, Theorem 3, is shown to contain as special cases well-known theorems of Ostrowski, Pólya, V. Bernstein, and others; these theorems assert, as a consequence of properties of various densities or of the index of condensation of a sequence of positive exponents, that there are singular points near to or on certain segments of the axes of convergence or holomorphy. Somewhat arbitrarily, two of these special cases are labeled here as theorems. Theorem 4 is known; it is due to Leont'ev [7, p. 83], but a remark in a footnote of an earlier paper of Pólya [16] indicates that the result was probably known to him, also. It is a generalization of the gap theorem proved for Taylor series by Fabry and for Dirichlet series with positive exponents by Carlson and Landau [3] and by Szász [21]. In this case, $D^* = 0$, and therefore it makes no difference whether the disc or indicator diagram is used; for each consists of a single point. Theorem 5 seems to be new. Its conclusion, very roughly speaking, is that if the reflected indicator diagram of a certain even entire function which is generated by the exponents is "placed on top of" a frontier point of the region of convergence, then it contains a singular point for the sum of the series.

We conclude our applications of Mandelbrojt's inequality with the following generalization of theorems of Liouville and Mandelbrojt. If the sum $f(z)$ of the series is bounded "near" any line and also in an angle, however small, which is bisected by this line, then $f(z)$ vanishes identically.

The present form of Mandelbrojt's inequality is a result of the author's attempts to generalize Jentzsch's theorem, Ostrowski's overconvergence theorem, and similar results pertaining to the behavior of partial sums of the series near the boundary of the region of convergence. While this program met with some success, it eventually became clear that Mandelbrojt's inequality is a more powerful tool than is needed for Jentzsch's theorem; on the other hand, while the older form of the inequality does not suffice for the proof of Ostrowski's theorem because one can not "approach closely enough" to the boundary, even the present form requires hypotheses which are not really essential. These considerations lead to another estimate of the coefficients of the series. Its proof requires only the author's generalization [4] of the Cauchy-Hadamard formula. This inequality and some of its applications, indicated at the beginning of this paragraph, conclude the paper.

The results given here are in essence a part of the author's thesis presented to The Rice Institute in 1954. A generalization to asymptotic series is to be found in the thesis of Martin Wright, presented there in 1956; and the author has profited from discussions with Professor Wright.

The author would like very much to express his gratitude for the inspiration, encouragement, and guidance which he has received from Professor Mandelbrojt.

2. **Continuation by means of a compact set.** Given a real number k and two sets A and B in the complex plane, we shall denote by $A+kB$ the set

$$\{z: z = a + kb, a \in A, b \in B\}.$$

Also we shall denote by E the unit disc $|z| \leqq 1$; but we shall not attempt to distinguish between a point z_0 and the set consisting of the single point z_0. Then, for example, z_0+A denotes the translate of the set A through a distance $|z_0|$ in the direction arg z_0; and $A+kE$ is the union of all discs $|z-a| \leqq k$, where a is a member of A. This familiar addition of sets is associative and commutative, of course.

Suppose $f(z)$ is a uniform (i.e. single-valued) analytic function, and let its domain of existence be denoted by D. Let there be given a compact set C. Then, given any two points z_1 and z_2, not necessarily in D, we will say that $f(z)$ can be continued analytically by means of C from z_1+C to z_2+C if there exists any path joining z_1 and z_2 such that $z+C$ is contained in D for every point z on this path.

For example, if D is the half-plane $x>1$ and C is the closed disc $2+E$, then $f(z)$ can be continued analytically by means of C from $\epsilon+C$ to $\epsilon+i+C$ if $\epsilon>0$, but not if $\epsilon<0$.

The assumption that $f(z)$ is uniform, which will be adhered to throughout, is made only in order to avoid cumbersome statements involving the Riemann surface of $f(z)$. Extensions of the results of this paper to more general situations should be obvious. In these situations one pictures the set C moving along "beside" or "on top of" (depending on whether the origin lies outside of or on the set C) the path joining z_1 and z_2 without "striking a singular point" of the branch in question of $f(z)$.

3. **Mandelbrojt's inequality.** It is known (see for example [4]) that, as a consequence of (1.3), if the series (1.1) converges in an open disc to a function $f(z)$, then the convergence is uniform on every compact subset of the disc, and $f(z)$ is analytic there. Thus it will be meaningful to speak in the following theorem of the analytic continuation of the function to which the series is assumed to converge. Now the domain of convergence is supposed, in our theorem, to contain a translate of the reflection C with respect to the imaginary axis of the indicator diagram of an entire function $g(z)$ which is of exponential type and which has simple zeros at the points $\lambda_1, \lambda_2, \cdots$. (It will be assumed that the reader is familiar with the notions of indicator, indicator diagram, conjugate diagram and Borel transform and of their basic properties, which can be found in Chapter 5 of [2].) Let τ be the type of $g(z)$; i.e.

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r},$$

where $M(r)$ is the maximum modulus function of $g(z)$. It is well known that the set C is a subset of the disc $|z| \leqq \tau$, and that C has at least one point in common with the circle $|z| = \tau$. However, it would be well to have in mind an example of such an entire function $g(z)$, and that which is best known is the function

$$\prod_{n=1}^{\infty} (1 - z^2/\lambda_n^2). \tag{3.1}$$

We shall refer to it as the even entire function generated by the sequence $\{\lambda_n\}$. In the case of this function, we can be moderately precise (taking into consideration the weak hypotheses on the sequence $\{\lambda_n\}$) about the location of C. Let

$$\overline{D}(r) = \frac{1}{r}\int_0^r \frac{n(x)}{x}\,dx,$$

where $n(x)$ is the number of λ's with modulus less than x, and let

$$\overline{D}^* = \limsup_{r\to\infty} \overline{D}(r).$$

Then $D^*/e \leqq \overline{D}^* \leqq D^*$ [9, p. 53]; and an example shows that this statement becomes false if e is replaced by a smaller number. Now (3.1) is known [9, p. 58] to be an entire function of exponential type $\tau \leqq \pi \overline{D}^*$, and Jensen's formula,

$$r\overline{D}(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \left| g(re^{i\theta}) \right| d\theta,$$

shows that $\overline{D}^* \leqq \tau$. Thus, in summary: in the case of the even entire function (3.1) generated by $\{\lambda_n\}$, we have

$$D^*/e \leqq \overline{D}^* \leqq \tau \leqq \pi\overline{D}^* \leqq \pi D^*.$$

Of course, if the limit (1.5) exists, then $\tau = \overline{D}^* = D^* = D$.

THEOREM 1. *Let* $\{\lambda_n\}$ *be a sequence of complex numbers satisfying conditions* (1.2) *and* (1.3); *let* $g(z)$ *be an entire function which is of exponential type and has simple zeros at the points* $\lambda_1, \lambda_2, \cdots$, *and let* C *be the reflection, with respect to the imaginary axis, of the indicator diagram of* $g(z)$. *Suppose that a Dirichlet series* (1.1) *converges to a uniform function* $f(z)$ *in a domain containing a translate* $z_1 + C$ *of the set* C.

Assume, furthermore, that $f(z)$ *can be continued analytically by means of* C *from* z_1+C *to some set* z_2+C. *Let* $\epsilon > 0$ *be sufficiently small so that* $f(z)$ *is analytic on the set* $z_2+C+\epsilon E$, *and let* m *denote the maximum of* $|f(z)|$ *on this set. Then there exists a constant* k, *depending only on* ϵ *and* $g(z)$, *such that*

$$|c_n| \leqq km\left|\frac{e^{\lambda_n z_2}}{g'(\lambda_n)}\right|. \tag{3.2}$$

Proof. Let

$$g_n(z) = \frac{g(z)}{z - \lambda_n};$$

this is an entire function that clearly has the same indicator function (and hence the same indicator diagram) as $g(z)$. Denote by $b_n(z)$ the Borel transform of $g_n(z)$. By hypothesis there exists a path L joining z_1 and z_2 such that $f(z)$ is analytic on $z+C$ if z is on L. It follows that there exists a number $\eta > 0$ such that $f(z)$ is analytic on $z+C+\eta E$ if z is on L. Let

$$\phi_n(z) = \frac{1}{2\pi i}\oint f(z - t) b_n(t)\,dt \tag{3.3}$$

where the path of integration is the boundary Γ_η of $-C+\eta E$. Now $b_n(t)$ is known to be analytic outside of the conjugate diagram of $g_n(z)$ and hence on Γ_η and, given any point of L, if z belongs to a sufficiently small neighborhood of this point, then $f(z-t)$ is an analytic function of t on Γ_η. Thus the function $\phi_n(z)$ is well defined in a domain containing L by the representation (3.3), and it is analytic in this domain. For all z in a sufficiently small neighborhood of z_2, the number η may be replaced, in this representation of $\phi_n(z)$, by the number ϵ of the theorem. In other words, Γ_η can be replaced by Γ_ϵ, the boundary of $-C+\epsilon E$. It is clear that if we denote by $\|\Gamma_\epsilon\|$ the length of Γ_ϵ, and by $\|b_n\|$ the maximum modulus of $b_n(z)$ on Γ_ϵ, then

$$|\phi_n(z_2)| \leqq \frac{1}{2\pi} m\|b_n\|\|\Gamma_\epsilon\|.$$

We will show that the numbers $\|b_n\|$ are bounded and hence that there exists a constant k, which depends only on ϵ and $g(z)$, such that

$$|\phi_n(z_2)| \leqq km \tag{3.4}$$

for $n=1, 2, \cdots$.

Let R_n be the ray, issuing from the origin, on which $\lambda_n z = -|\lambda_n z|$; and let L_n be a line which is perpendicular to R_n and which passes through a point z_n on Γ_ϵ such that

$$|b_n(z_n)| = \|b_n\|.$$

Because of the convexity of the region $-C+\epsilon E$, it is possible to move continuously from z_n along a segment of L_n and then along one quarter of the circumference of the circle whose center is at the intersection of R_n with L_n and whose radius is $2(\tau+\epsilon)$, where τ is the type of $g(z)$, and then along R_n to $z=\infty$ without intersecting the interior of the region $-C+\epsilon E$. Let the path just described be denoted by P_n; let the part of it consisting of a finite line segment and a quarter of a circle be denoted by P_n'; and let the remaining segment, whose finite endpoint we shall call w_n, be denoted by P_n''.

Now, when the real part of z is sufficiently large, the Borel transform of $g(z)$ has the representation

$$\begin{aligned} b(z) &= \int_0^{+\infty} e^{-zx} g(x)dx \\ &= \int_0^{+\infty} e^{-zx}(x-\lambda_n)g_n(x)dx \\ &= -b_n'(z) - \lambda_n b_n(z). \end{aligned}$$

Introducing the integrating factor $\exp(\lambda_n z)$, we see that

$$\int_{z_n}^{\infty} e^{\lambda_n t} b(t)dt = e^{\lambda_n z_n} b_n(z_n),$$

where the path of integration is P_n. Thus

$$\|b_n\| = \left| \int_{z_n}^{\infty} e^{\lambda_n (t-z_n)} b(t)dt \right|.$$

Regarding this integral as the sum of integrals over P_n' and P_n'', and applying the principal of the maximum to $b(z)$, we note that

$$\left|\int_{w_n}^{\infty} e^{\lambda_n(t-z_n)}b(t)dt\right| \leqq \|b\| e^{|\lambda_n z_n|}\int_{|w_n|}^{+\infty} e^{-|\lambda_n|x}dx$$
$$= \frac{\|b\|}{|\lambda_n|} e^{|\lambda_n|(|z_n|-|w_n|)} \leqq \frac{\|b\|}{|\lambda_1|},$$

where $\|b\|$ is the maximum of $|b(z)|$ on Γ_ϵ, and that the integral over P_n' is bounded by the product of $\|b\|$ and the length of P_n', since the real part of $\lambda_n(t-z_n)$ is nonpositive on this path. Thus

$$\|b_n\| \leqq \|b\|\left[\frac{1}{|\lambda_1|} + 4\pi(\tau+\epsilon)^2\right],$$

which establishes (3.4).

To complete the proof of the theorem, we replace $f(z-t)$ by its series representation, which, for z in a sufficiently small neighborhood of z_1, is uniformly convergent with respect to t on Γ_ϵ. Thus it is permissible to integrate term-by-term, and we obtain

$$\phi_n(z) = \sum_{r=1}^{\infty} c_r e^{-\lambda_r z}\left[\frac{1}{2\pi i}\oint e^{\lambda_r t} b_n(t)dt\right].$$

But the expression in brackets is well known to be $g_n(\lambda_r)$, which is zero unless $r=n$. Thus

(3.5) $$\phi_n(z) = c_n g_n(\lambda_n) e^{-\lambda_n z}.$$

Since

$$g_n(\lambda_n) = \lim_{z\to\lambda_n} \frac{g(z)-0}{z-\lambda_n} = g'(\lambda_n),$$

(3.4) and (3.5) imply (3.2).

4. **A convergence theorem.** It is a familiar fact that if the exponents $\{\lambda_n\}$ are positive, then the series (1.1) converges uniformly on every angular region $|\arg(z-z_0)| \leqq \gamma < \pi/2$ contained in the interior of the region of convergence of the series. This is no longer true if the arguments of the exponents are arbitrary, as shown by the example: $c_n = e^{-n}$, $\lambda_n = (i)^n$. Here the series converges for every z, but the convergence is not uniform on any unbounded set since, given any n and then any z with $|z|$ sufficiently large,

$$|c_r e^{-\lambda_r z}| > 1$$

for at least one of the four values $r=n, n+1, n+2, n+3$. The reason for the failure of the methods traditionally applied to convergence questions for

Dirichlet series with positive exponents is that these methods depend on the use of Abel's transformation, which appears to be useless in the case of complex exponents. However, the following theorem gives sufficient conditions, without restricting the arguments of the exponents, for uniform convergence of the series on a possibly unbounded set U.

THEOREM 2. *Assume that the conditions stated in the first paragraph of Theorem* 1 *are satisfied.*

Assume, furthermore, that there is given a set U, and that $f(z)$ can be continued analytically by means of C from z_1+C to each set $u+C$ such that u belongs to U. If

$$h = \max\left(0, \limsup_{n\to\infty} |\lambda_n|^{-1} \log |g'(\lambda_n)|^{-1}\right) < \infty, \tag{4.1}$$

and if there exists a number $\eta>0$ such that $f(z)$ is regular and bounded on $U+C+(h+\eta)E$, then the series (1.1) *converges uniformly on the set U.*

Proof. For any point z_2 in $U+(h+\eta/2)E$, we can take the number ϵ in Theorem 1 to be $\eta/2$ and conclude that the inequality (3.2) is valid if we take for m in this inequality a bound for $|f(z)|$ on $U+C+(h+\eta)E$. Now let u be any point of U, and then choose for z_2 the point

$$z_2 = u + (h + \eta/2) \exp\{i(\pi - \arg \lambda_n)\}.$$

It follows that, if n is sufficiently large so that

$$|\lambda_n|^{-1} \log |g'(\lambda_n)|^{-1} \leqq h + \eta/3,$$

then

$$\begin{aligned} |c_n e^{-\lambda_n u}| &\leqq km \exp\{\Re[\lambda_n(z_2 - u)] + (h + \eta/3)|\lambda_n|\} \\ &= km \exp\{-\eta/6|\lambda_n|\}. \end{aligned}$$

Since (1.3) implies the existence of a number $\theta>0$ such that $|\lambda_n|>n\theta$ for $n=1, 2, \cdots$, uniform convergence on U has been proved.

The number h is, of course, very closely related to the index of condensation, as defined by V. Bernstein [1, pp. 25–27], of the sequence $\{\lambda_n\}$. One difference between the two numbers is that h appears to depend on the choice of the entire function $g(z)$. However, if the index of condensation is non-negative, if the sequence of exponents has a density, and if we choose for $g(z)$ the even entire function (3.1) generated by the sequence, then h clearly is the index of condensation.

5. **Detection of singularities.** The following corollary of Theorem 2 throws some light on the relationship between the index of condensation and the various densities which play a role in the known theorems on detection of singularities.

THEOREM 3. *Assume that the conditions stated in the first paragraph of Theorem* 1 *are satisfied.*

Assume, furthermore, that $f(z)$ *can be continued analytically by means of* C *from* z_1+C *to a set* $b+C$, *where* b *is a frontier point of the region of convergence. If condition* (4.1) *is satisfied, then* $f(z)$ *has at least one singular point on* $b+C+hE$.

Proof. We need only observe that if the conclusion of the theorem were false, then we could take the set U in Theorem 2 to be a sufficiently small neighborhood of b and conclude that the series converges throughout U, which is impossible.

The conclusion of Theorem 3 can be made much more explicit by imposing restrictions on the exponents in such a way as to reduce the size of the number h or of the set C, or to reduce the curvature of the boundary of the region of convergence. Some examples of this follow.

Mandelbrojt [9, p. 61] has shown that if condition (1.4) is satisfied and if $g(z)$ is the even entire function (3.1) generated by the sequence $\{\lambda_n\}$, then

$$h \leqq 3[3 - \log (pD^*)]D^*.$$

Thus, if in Theorem 3 we replace condition (1.3) by (1.4), then, since C for this particular $g(z)$ is contained in πD^*E, Theorem 3 becomes a generalization of a well known theorem of Ostrowski [12] which has been made more precise by Mandelbrojt [9, p. 268]. This theorem "approaches as a limit" as D^* approaches zero the following theorem of Leont'ev [7, p. 83].

THEOREM 4. *If* $\{\lambda_n\}$ *is a sequence of complex numbers satisfying* (1.4) *and also* $D^*=0$ (*hence* $D=0$), *and if a Dirichlet series* (1.1) *converges to a function* $f(z)$ *in some open set, then the domain of existence of this* (*necessarily uniform*) *function* $f(z)$ *is the* (*convex*) *interior of the region of convergence of the series.*

In order that $h=0$ it is not necessary, however, that $D=0$. Levinson, generalizing a result of V. Bernstein, has shown that $h=0$ if the density D merely exists, provided (1.4) is satisfied [2, p. 186]. This fact gives the following modification of Theorem 3.

THEOREM 5. *Let* $\{\lambda_n\}$ *be a sequence of complex numbers, having a density* D *given by* (1.5) *and satisfying* (1.4); *let* $g(z)$ *be the even entire function* (3.1) *generated by this sequence; and let* C *be the reflection, with respect to the imaginary axis, of the indicator diagram of* $g(z)$. *Suppose that a Dirichlet series* (1.1) *converges to a uniform function* $f(z)$ *in a domain containing a translate* z_1+C *of the set* C. *Then it is not possible to continue* $f(z)$ *analytically by means of* C *from* z_1+C *to any set* $b+C$, *where* b *is a frontier point of the region of convergence.*

We can reduce the size of C and at the same time reduce the curvature of the boundary of the region of convergence by assuming that the exponents are positive; for it follows from a theorem of Carlson [2, p. 137] that if the exponents are positive and have a density D, then the indicator diagram of the even entire function (3.1) generated by the sequence is the vertical line segment joining the points πDi and $-\pi Di$. Other hypotheses are known to lead to this same conclusion [6, pp. 87, 89]. With these hypotheses, Theorem 3 gives as an immediate consequence a number of theorems of V. Bernstein [1, pp. 134–140], the most important of his results being the following. The difference between the abscissas of convergence and of holomorphy is not greater than h; every interval of length $2\pi D$ of the axis of holomorphy contains at least one singular point of the direct analytic continuation of the function defined by the series. Furthermore, Theorem 5 (which is, of course, itself a special case of Theorem 3) becomes the famous theorem of Pólya which states that if the exponents are positive and have a density D and satisfy condition (1.4), then every segment of length $2\pi D$ of the axis of convergence contains a singular point of the direct analytic continuation of the sum of the series. These results clearly remain valid even if the sequence $\{\lambda_n\}$ does not have a density, provided it is a subsequence of a sequence $\{\mu_n\}$ which has a density D and to which there corresponds a suitable value of h (finite or zero, depending on the application) defined with respect to the even entire function (3.1) generated by the sequence $\{\mu_n\}$; for we can take this even entire function as the function $g(z)$ in Theorem 3. Furthermore, Bernstein [1, pp. 289–293] has shown the existence of a sequence $\{\mu_n\}$ "containing" the sequence $\{\lambda_n\}$ and having a density equal to D and having the same index of condensation as $\{\lambda_n\}$, provided that $\{\lambda_n\}$ is a sequence of real numbers with a finite maximum density equal to D, as defined by Pólya [1, p. 22; 17].

6. **A uniqueness theorem.** The last application of the fundamental inequality is the following generalization of theorems of Liouville and Mandelbrojt [9, p. 253]. It can, of course, be rephrased as a uniqueness theorem.

THEOREM 6. *Let L be any line passing through the origin, let A be any angular opening (no matter how small) with vertex at the origin and bisected by L, let K be the exterior of any disc, and let*

$$R = (L \cup A) \cap K.$$

Assume that the conditions stated in the first paragraph of Theorem 1 are satisfied, and that $f(z)$ can be continued analytically by means of C from the set z_1+C to each set $r+C$ such that r belongs to R. If there is a number $\epsilon>0$ such that $f(z)$ is regular and bounded on the set $R+C+\epsilon E$, then $f(z)\equiv 0$.

Proof. For every n there is a ray, $\arg z=\alpha$, in $L\cup A$ such that

$$\theta_n = |\alpha - (\pi - \arg \lambda_n)| < \pi/2.$$

If z_2 is on this ray and $|z_2|$ is sufficiently large, then the inequality (3.2) is valid with km independent of z_2. Thus

$$\begin{aligned} | c_n | &\leqq km | g'(\lambda_n) |^{-1} \exp \{ | \lambda_n z_2 | \cos (\arg \lambda_n + \alpha) \} \\ &= km | g'(\lambda_n) |^{-1} \exp \{ -(| \lambda_n | \cos \theta_n) | z_2 | \}. \end{aligned}$$

Since $|z_2|$ can be taken arbitrarily large, we conclude that $c_n = 0$.

7. **Another estimate of the coefficients.** An estimate will now be given which has something of the appearance of Mandelbrojt's inequality and has much simpler hypotheses. It is useful provided one "stays inside" the region of convergence, or else is satisfied with a relatively crude estimate.

THEOREM 7. *If R is a compact subset of the interior of the region of convergence Δ of the series* (1.1), *and if δ denotes the distance from R to the frontier of Δ, then, for every $\epsilon > 0$, there exists a constant k such that*

$$| c_n | \leqq k e^{-|\lambda_n| (\delta - \epsilon)} | e^{\lambda_n z} | \tag{7.1}$$

for all z on R.

Proof. Let

$$d(n, z) = - | \lambda_n |^{-1} \log | c_n e^{-\lambda_n z} | .$$

Since the distance from R to the boundary of the convex region D on which

$$d(z) = \liminf_{n \to \infty} d(n, z) \geqq 0$$

is not less than δ, it follows [4] that $d(z) \geqq \delta$ on R. Thus for each z_α on R there is a number n_α such that $d(n, z_\alpha) \geqq \delta - \epsilon/2$ for $n > n_\alpha$. But then, from the geometric interpretation of $d(n, z)$ as the signed distance from z to a certain line, $d(n, z) \geqq \delta - \epsilon$ if z belongs to the disc D_α which has its center at z_α and is of radius $\epsilon/2$, provided $n > n_\alpha$. Thus there is a constant k_α such that

$$\begin{aligned} | c_n e^{-\lambda_n z} | &= e^{-|\lambda_n| d(n,z)} \\ &\leqq k_\alpha e^{-|\lambda_n| (\delta - \epsilon)} \end{aligned}$$

for z on D_α and every n. Since a finite number of these discs D_α cover R, the theorem is proved.

8. **Jentzsch's theorem.** Theorem 7 will now be applied in the proof, suggested by a method due to Montel [10], of a generalization of Jentzsch's theorem.

THEOREM 8. *If $\{\lambda_n\}$ is a sequence of complex numbers satisfying condition* (1.4) *and if a Dirichlet series* (1.1) *has a region of convergence Δ whose interior is not empty, then every frontier point of Δ is a limit point of zeros of the partial sums*

$$s_n(z) = \sum_{r=1}^{n} c_r e^{-\lambda_r z}.$$

Proof. Suppose, if possible, that there is a frontier point b of Δ which is the center of an open disc C that is free of zeros of the functions $s_n(z)$. (We can obviously ignore any finite number of zeros.) Let

$$h_n(z) = s_n(z)^{1/|\lambda_n|},$$

by which we mean a branch such that $h_n(z)$ is real when $s_n(z)$ is real. Now it follows from (7.1), if we choose for z merely any interior point of Δ, that there exists a number m such that

$$|c_n| \leqq ke^{m|\lambda_n|}.$$

Thus, if $\rho>0$, then for $|z| \leqq \rho$,

$$\begin{aligned} |s_n(z)| &\leqq k \sum_{r=1}^{n} e^{(m+\rho)|\lambda_r|} \\ &= ke^{(m+\rho)|\lambda_n|} \sum_{r=1}^{n} \exp\{(m+\rho)(|\lambda_r| - |\lambda_n|)\} \\ &\leqq ke^{(m+\rho)|\lambda_n|} \sum_{r=0}^{\infty} e^{-(m+\rho)pr}, \end{aligned}$$

which implies that the family of functions $\{h_n(z)\}$ is bounded, and hence normal, in the disc C. But, from the convexity of the closure of Δ, the disc C contains a compact set K of interior points of Δ on which the sequence $\{s_n(z)\}$ converges uniformly to a function which is bounded away from zero on K. It follows that the functions $\{h_n(z)\}$ converge uniformly on K, and hence on any compact subset of C, to the function identically equal to 1. Now, as a consequence of (1.4), the interior of Δ coincides with that of the convex set D described in the preceding section. Let S be any compact subset of C such that S has no point in common with D. For any $\epsilon>0$, $|h_n(z)| \leqq 1+\epsilon$ on S for all n sufficiently large, and hence

$$|c_n e^{-\lambda_n z}| = |s_n(z) - s_{n-1}(z)| \leqq 2(1+\epsilon)^{|\lambda_n|},$$

that is,

$$d(n, z) \geqq -\log(1+\epsilon) - |\lambda_n|^{-1} \log 2$$

for these values of z and n. Thus $d(z) \geqq -\log(1+\epsilon)$ on S. Since ϵ is arbitrary, $d(z) \geqq 0$ on S, which implies that S is a subset of D. This is a contradiction, and the theorem is proved.

9. **Ostrowski's theorem.** The following over-convergence theorem generalizes that of Ostrowski, which, in turn, is well known to give Hadamard's gap theorem as an immediate consequence. The proof here is actually closer, because of the use of Theorem 7, to that originally given by Ostrowski [11] than to slightly different later versions such as Bernstein's proof for Dirichlet

series with positive exponents [1, p. 39], although all of these are based on Hadamard's three-circle theorem.

It will be assumed that at some point b of the boundary of the region of convergence Δ there is a disc which is tangent to the boundary there and whose interior is contained in the interior of Δ. A sufficient condition for this to be the case can be found in [20]. However, a sufficient condition which appears to be easier to apply is the following. Suppose the point b is an interior point of an arc of the boundary which has a representation

$$z(t) = x(t) + iy(t)$$

for $t_1 \leqq t \leqq t_2$, with $x''(t)$ and $y''(t)$ continuous and

$$[x'(t)]^2 + [y'(t)]^2 > 0.$$

Then a disc of the sort described exists, for there is no loss of generality in supposing that the parameter is arc length s with $b = z(0)$, and that b is the origin and that the tangent there is horizontal. We need only show that for some $r > 0$ and every s sufficiently small in absolute value,

$$\phi(s) = [x(s)]^2 + [y(s) - r]^2 - r^2 \geqq 0;$$

for the first two terms represent the square of the distance from $z(s)$ to the center ir of a disc which is tangent to the boundary at the origin. But, since $\phi(0) = \phi'(0) = 0$, $\phi(s) = s^2\phi''(\xi)/2$, where ξ lies between 0 and s. An easy computation shows that $\phi''(s) > 0$ when $|s| < 1/4M$, if $r = 1/2M$ and M is a bound for $x''(s)$ and $y''(s)$. (We are assuming here that Δ has a nonempty interior and a convex closure.)

THEOREM 9. *Given a sequence of complex numbers* $\{\lambda_n\}$ *satisfying condition* (1.4), *suppose there exists a number* $\theta > 0$ *and an infinite subsequence* $\{n_i\}_{i=1}^{\infty}$ *of the positive integers such that*

$$\frac{|\lambda_{n_i+1}|}{|\lambda_{n_i}|} \geqq 1 + \theta. \tag{9.1}$$

Suppose, furthermore, that a Dirichlet series (1.1), *having the numbers* $\{\lambda_n\}$ *as exponents, has a region of convergence* Δ *with a boundary point b at which there is an open disc tangent to the boundary and contained in the interior of* Δ. *If the sum of the series f(z) has a direct analytic continuation into a disc with center at b, then the partial sums*

$$s_i(z) = \sum_{r=1}^{n_i} c_r e^{-\lambda_r z}$$

converge uniformly in some neighborhood of b as $i \to \infty$.

Proof. We shall select a point z_1 in the interior of Δ and on the normal at b to the boundary. Let $a = |z_1 - b|$. Then we shall denote by C_1, Γ, C_2, and C_3 the four closed discs with centers each at z_1 and radii, respectively, of $a-a^2$, $a-a^3$, $a+a^3$, and $a+a^2$. The theorem will be proved by showing that the functions $s_i(z)$ converge uniformly on C_2; but to do this it will be necessary to assume that a is taken sufficiently small so that

1. $a<1$,

2. $$g(a,\theta) = (1+\theta)(a^3-a^2)\log\frac{1+a}{1+a^2} + (a^3+a^2)\log\frac{1+a^2}{1-a}$$
$$= -\theta a^3 + 5(1+\theta/2)a^4 + \cdots$$
$$< -\theta a^3/2,$$

3. $f(z)$ is analytic on C_3,

and

4. Γ belongs to the interior of Δ.

It is clear that z_1 can be chosen in this way.

Now (7.1) shows that there is a number k_1 such that

$$|c_n| \leqq k_1 |e^{\lambda_n \gamma}|,$$

where γ is any point on Γ; and thus for z on C_3, since there is always a γ on Γ such that $|\gamma - z| \leqq a^2+a^3$,

$$|c_n e^{-\lambda_n z}| \leqq k_1 e^{(a^2+a^3)|\lambda_n|}.$$

Then, from a computation in the proof of Theorem 8 and the fact that $f(z)$ is bounded on C_3, it follows that there exists a k_2 such that

$$|f(z) - s_i(z)| \leqq k_2 e^{(a^2+a^3)|\lambda_{n_i}|}$$

for $i=1, 2, \cdots$ and z on C_3.

On the other hand, if z lies on C_1, there are points $\{\gamma_n\}_{n=1}^{\infty}$ on the circumference of Γ such that

$$\arg(\gamma_n - z) = \pi - \arg \lambda_n$$

and hence

$$\Re[\lambda_n(\gamma_n - z)] = -|\lambda_n(\gamma_n - z)| \leqq -|\lambda_n|(a^2-a^3).$$

Thus on C_1

$$\left|\sum_{r=n+1}^{\infty} c_r e^{-\lambda_r z}\right| \leqq k_1 \sum_{r=n+1}^{\infty} e^{-|\lambda_r|(a^2-a^3)}$$
$$\leqq k_1 \exp\{-|\lambda_{n+1}|(a^2-a^3)\} \sum_{r=0}^{\infty} e^{-p(a^2-a^3)r}$$
$$= k_3 \exp\{-|\lambda_{n+1}|(a^2-a^3)\}.$$

In particular

$$|f(z) - s_i(z)| \leqq k_3 \exp\{-(a^2 - a^3)(1 + \theta)|\lambda_{n_i}|\}$$

on C_1.

Let us denote the maximum modulus of $f(z)-s_i(z)$ on C_1, C_2, and C_3 by M_1, M_2, and M_3, respectively. Then, from the three circle theorem,

$$\begin{aligned}\log\frac{a+a^2}{a-a^2}\log M_2 &\leqq \log\frac{a+a^2}{a+a^3}\log M_1 + \log\frac{a+a^3}{a-a^2}\log M_3 \\ &\leqq \log\frac{1+a}{1+a^2}\left[\log k_3 - (a^2-a^3)(1+\theta)|\lambda_{n_i}|\right] \\ &\quad + \log\frac{1+a^2}{1-a}\left[\log k_2 + (a^2+a^3)|\lambda_{n_i}|\right] \\ &= g(a,\theta)|\lambda_{n_i}| + k_4 \\ &\leqq -\theta a^3|\lambda_{n_i}|/2 + k_4,\end{aligned}$$

where k_4 is independent of i. Thus

$$\lim_{i\to\infty}\log M_2 = -\infty,$$

which completes the proof.

Piranian [13] has shown that condition (9.1) can be relaxed in such a way that it still implies convergence of the partial sums $s_i(z)$ on, but no longer outside of, the circle of convergence of a power series. The method of proof given above can be extended to generalize Piranian's theorem to Dirichlet series with complex exponents; but the details will not be given here, since they are more complicated than the computations in Piranian's paper, which are not simple.

References

1. V. Bernstein, *Leçons sur les progrès récents de la théorie des séries de Dirichlet*, Paris, 1933.

2. R. P. Boas, Jr., *Entire functions*, New York, 1954.

3. F. Carlson and E. Landau, *Neuer Beweis und Verallgemeinerungen des Fabryschen Lückensatzes*, Nachr. Ges. Wiss. Göttingen (1921) pp. 184–188.

4. T. M. Gallie, Jr., *Region of convergence of Dirichlet series with complex exponents*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 627–629.

5. E. Hille, *Note on Dirichlet's series with complex exponents*, Ann. of Math. vol. 25 (1924) pp. 261–278.

6. J. P. Kahane, *Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par des sommes d'exponentielles*, Ann. Inst. Fourier, Grenoble vol. 5 (1953–1954) pp. 39–130.

7. A. F. Leont'ev, *Ryady polinomov Dirichlet i ih obobščeniya*, Trudy Mat. Inst. Steklov, vol. 39, Izdat. Akad. Nauk SSSR, Moscow, 1951.

8. S. Mandelbrojt, *Analytic continuation and infinitely differentiable functions*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 239–248.

9. ———, *Séries adhérentes, regularisation des suites, applications*, Paris, 1952.

10. P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Paris, 1927.

11. A. Ostrowski, *Über eine Eigenschaft gewisser Potenzreihen mit unendlich vielen verschwindenden Koeffizienten*, Preuss. Akad. Wiss. Sitzungsber. (1921) pp. 557–565.

12. ———, *Einige Bemerkungen über Singularitäten Taylorscher und Dirichletscher Reihen*, Preuss. Akad. Wiss. Sitzungsber. (1923) pp. 39–44.

13. G. Piranian, *On the convergence of certain partial sums of a Taylor series with gaps*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 881–885.

14. G. Pólya, *Analytische Fortsetzung und konvexe Kurven*, Math. Ann. vol. 89 (1923) pp. 179–191.

15. ———, *Über die Existenz unendlich vieler singulärer Punkte auf der Konvergenzgeraden gewisser Dirichletscher Reihen*, Preuss. Akad. Wiss. Sitzungsber. (1923) pp. 45–50.

16. ———, *Eine Verallgemeinerung des Fabryschen Lückensatzes*, Nachr. Ges. Wiss. Göttingen (1927) pp. 187–195.

17. ———, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Math. Z. vol. 29 (1929) pp. 549–640.

18. J. F. Ritt, *On a general class of linear homogeneous differential equations of infinite order with constant coefficients*, Trans. Amer. Math. Soc. vol. 18 (1917) pp. 27–49.

19. ———, *Note on Dirichlet series with complex exponents*, Ann. of Math. vol. 26 (1924) p. 144.

20. W. Seidel, *Über die Ränderzuordnung bei konformen Abbildungen*, Math. Ann. vol. 104 (1930) pp. 182–243.

21. O. Szász, *Über Singularitäten von Potenzreihen und Dirichletschen Reihen am Rande des Konvergenzbereiches*, Math. Ann. vol. 85 (1922) pp. 99–110.

22. V. Väisälä, *Verallgemeinerung des Begriffes der Dirichletschen Reihen*, Acta Univ. Dorpatensis (A) vol. I: 2 (1921) pp. 1–32.

23. G. Valiron, *Sur les solutions des équations différentielles linéaires d'ordre infini et à coefficients constants*, Ann. École Norm. (3) vol. 46 (1929) pp. 25–53.

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